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Interpolating Constrained Control of Interconnected Systems

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Abstract: This paper presents a decentralised interpolating control scheme for the control of linear discrete-time interconnected systems with local state and control constraints. The control law of each distinct subsystem relies on the gentle interpolation between a local high-gain controller, which satisfies some user-desired performance specifications, with a global low-gain controller. For each subsystem both low- and high-gain controllers can be efficiently determined off-line, while the inexpensive interpolation between them is performed on-line. For the interpolation, a new low-dimensional linear programming formulation is developed, which is computationally less expensive compared to previous works. Therefore, it is appropriate for real-time control of large-scale interconnected systems. Proofs of recursive feasibility and asymptotic stability of the interpolating scheme are given for decoupled as well as interconnected subsystems with coupled state constraints. Two numerical examples show that the proposed decentralised interpolating control outperforms previously proposed interpolating schemes. Finally, it is faster than model predictive control, while their control behaviour and performance is almost identical.

Keywords: Invariant Sets; Interpolation; Interconnected Systems; Decentralised Control;

1. INTRODUCTION

Interpolating Control (IC) is a novel approach that incorporates the state and control constraints into the control problem formulation and significantly reduces the computational effort (Nguyen, 2014) compared to optimisation-based schemes such as Model Predictive Control (MPC) (Kouvaritakis and Cannon, 2016). The main idea of IC is to blend a local high-gain controller, which satisfy some user-desired performance specifications, with a global low-gain vertex controller via interpolation. Although interpolation is appealing as an idea, its complexity is in direct relationship with the computational complexity of the low-gain vertex controller, which might be high for large-scale systems (Nguyen et al., 2013). Nguyen et al. (2014) have proposed an improved interpolating control method to reduce computational complexity. Precisely, the global outer controller is determined in an augmented state and control space and thus no vertex representation of the controllable invariant set is needed (Nguyen et al., 2014).

To overcome the computational complexity of the vertex controller, this work proposes a decentralised Interpolating Control (dIC) scheme to solve constrained control problems via distributed interpolation in low-dimensional spaces instead of for a large-scale system and guarantee stability and robustness. A feature of this approach is robustness that keeps the system stable despite any perturbations in the interconnections. Set invariance is important for interpolating control to guarantee recursive feasibility and asymptotic stability of the closed-loop system. This paper proposes to compute controlled invariant sets for local control design, which overcomes the computational burden of large-scale systems. A similar approach is pursued e.g. in Raković et al. (2010); Nilsson and Ozay (2016), where separable invariant sets are also computed. Moreover, computing the interpolating control for the whole system would be difficult because a low-gain high-dimension controller needs to be successfully computed. Alternatively, it is more convenient to determine local interpolating control for subsystems in a distributed way where possible interconnections are treated as additive bounded disturbances as proposed in this paper.

2. CONTROL PROBLEM FORMULATION

Consider a linear time-invariant interconnected dynamic system consisting of N subsystems

$$\mathcal{S}_i : \begin{cases} x_i(k+1) = A_i x_i(k) + B_i u_i(k) \\ \qquad\qquad + \sum_{j \in \mathcal{N}_i} e_{ij} \bar{A}_{ij} x_j(k), \quad i \in \mathcal{N}, \end{cases} \tag{1}$$

where $x_i(\cdot) \in \mathbb{R}^{n_i}$ and $u_i(\cdot) \in \mathbb{R}^{m_i}$ are, respectively, the (observable) state and control vectors for the subsystem $i \in \mathcal{N} = \{1, 2, \ldots, N\}$; $A_i \in \mathbb{R}^{n_i \times n_i}$ and $B_i \in \mathbb{R}^{n_i \times m_i}$ are the state and control matrices; and, $\bar{A}_{ij} \in \mathbb{R}^{n_i \times n_j}$ is an interconnection (adjacency) matrix between subsystem i and j, where $\mathcal{N}_i$ is the set of neighbour subsystems to i for information exchange; $e_{ij} \in [0, 1]$ are weighting constants, which model the strength of adjacent interconnections. If the adjacency matrices are null or $\mathcal{N}_j = \emptyset$, $\forall j \in \mathcal{N}$, then system (1) is *decoupled*. The overall system $\mathcal{S} = \bigcup_{i \in \mathcal{N}} \mathcal{S}_i$ involves a global state vector $x^\mathsf{T} = \begin{bmatrix} x_1^\mathsf{T} \; x_2^\mathsf{T} \; \cdots \; x_N^\mathsf{T} \end{bmatrix} \in \mathbb{R}^n$ and a global control vector $u^\mathsf{T} = \begin{bmatrix} u_1^\mathsf{T} \; u_2^\mathsf{T} \; \cdots \; u_N^\mathsf{T} \end{bmatrix} \in \mathbb{R}^m$, where $n = \sum_{i \in \mathcal{N}} n_i$ and $m = \sum_{i \in \mathcal{N}} m_i$.

The decentralised control problem of the interconnected system (1) is to design a controller that regulates each subsystem $i \in \mathcal{N} = \{1, 2, \ldots, N\}$ to the origin, where the i-th controller uses the local state vector $x_i(k)$ to generate the local control $u_i(k)$ for the plant. We assume that the state x_i is measurable and available for feedback in each subsystem, and that a state-feedback controller

$$u_i(k) = -K_i x_i(k), \quad i \in \mathcal{N} \tag{2}$$

exists such that $\mathcal{S} = \bigcup_{i\in\mathcal{N}} \mathcal{S}_i$ is stable, given some user-desired performance specifications. The resulting closed-loop state matrix $A_i - B_i K_i$, $i \in \mathcal{N}$, is Hurwitz.

For the constrained control problem, local state $x_i(k)$ and control $u_i(k)$ vectors are subject to polytopic constraints

$$\begin{cases} x_i(k) \in \mathcal{X}_i,\, \mathcal{X}_i = \{x_i \in \mathbb{R}^{n_i} \mid F_{x_i} x_i \leq g_{x_i}\}, \\ u_i(k) \in \mathcal{U}_i,\, \mathcal{U}_i = \{u_i \in \mathbb{R}^{m_i} \mid F_{u_i} u_i \leq g_{u_i}\}, \end{cases} \tag{3}$$

$\forall\, k \geq 0$, $i \in \mathcal{N}$, where F_{x_i}, F_{u_i} are constant matrices and g_{x_i}, g_{u_i} are constant vectors of appropriate dimension with positive elements, and the origin is contained in the interior of the sets. The inequalities are component-wise.

The decentralised control problem is a mean to obtain connective stability between subsystems, under structural perturbations (Šiljak, 1991). To account for couplings between subsystems, we consider an interconnected dynamical system with additive norm-bounded disturbances. Let $w_i = \sum_{j\in\mathcal{N}_i} e_{ij} \bar{A}_{ij} x_j(k)$, $i \in \mathcal{N}$, be the vector of interconnections. Perturbations due to couplings are then bounded by $\|w_i\| \leq \sum_{j\in\mathcal{N}_i} \|\bar{A}_{ij}\| \|x_j(k)\|$, $\forall\, i \in \mathcal{N}$, where $\|\cdot\|$ is the Frobenius norm and $x_j(k)$, $j \in \mathcal{N}_i$, is constrained by definition in (3). If a state x_j is free, a generous upper bound can be introduced to guarantee connective stability. Then the vector of interconnections may be brought to the general form of polytopic constraints

$$w_i(k) \in \mathcal{W}_i,\, \mathcal{W}_i = \{w_i \in \mathbb{R}^{n_i} \mid F_{w_i} w_i \leq g_{w_i}\}, \tag{4}$$

$\forall\, k \geq 0$, $i \in \mathcal{N}$, where F_{w_i} and g_{w_i} are suitable. Finally, the interconnected system (1) can be re-written as:

$$\mathcal{S}_i : x_i(k+1) = A_i x_i(k) + B_i u_i(k) + w_i(k), \quad i \in \mathcal{N}. \tag{5}$$

The system (5) paired with state, control, and disturbance constraints (3), (4) will be used as a basis for interpolating constrained control design in the next sections.

In the sequel, we provide some definitions from the invariant set theory that will be used in the rest of the paper (see e.g. Blanchini and Miani (2015); Borrelli et al. (2017)).

Definition 1. (Robust positively invariant set). Given the local controller (2) for each subsystem $i \in \mathcal{N}$, the set $\Omega_i \subseteq \mathcal{X}_i$ is a robust positively invariant constraint-admissible set with respect to $x_i(k+1) = (A_i - B_i K_i)\, x_i(k) + w_i(k)$ subject to the local constraints (3), (4), if and only if, $\forall\, x_i(k) \in \Omega_i$ and $\forall\, w_i(k) \in \mathcal{W}_i$, the system evolution satisfies $x_i(k+1) \in \Omega_i$ and $K_i x_i(k) \in \mathcal{U}_i$, $\forall\, k \geq 0$.

The largest robust positively invariant set that respects constraints is called *Maximal Admissible Set* (MAS) (Gilbert and Tan, 1991). The MAS can be defined as

$$\Omega_i = \{x_i \in \mathbb{R}^{n_i} : F_i^0 x_i \leq g_i^0\}, \quad i \in \mathcal{N}.$$

Definition 2. (Robust controllable invariant set). Given the interconnected system (5) and the constraints (3), (4), the set $\Psi_i \subseteq \mathcal{X}_i$ is robust controllable invariant, if and only if, for all $x_i(k) \in \Psi_i$, there exists an admissible control $u_i(k) \in \mathcal{U}_i$ such that $x_i(k+1) \in \Psi_i$, $\forall\, i \in \mathcal{N}$, $\forall\, w_i(k) \in \mathcal{W}_i$, $\forall\, k \geq 0$. The half-space representation of Ψ_i is given by

$$\Psi_i = \{x_i \in \mathbb{R}^{n_i} : F_i^1 x_i \leq g_i^1\}, \quad i \in \mathcal{N}.$$

Definition 3. (M-step controllable set). The set $P_i^M \subseteq \mathcal{X}_i$ is the set of all states for which exists an admissible robust control sequence such that the system (5) reaches the MAS Ω_i in no more than M steps along an admissible trajectory, i.e. one that satisfies (3), (4). The set P_i^M is called *M-step robust controllable set* and can be described by:

$$P_i^M = \{x_i \in \mathbb{R}^{n_i} : F_i^M x_i \leq g_i^M\}, \quad i \in \mathcal{N}.$$

Although iterative algorithms to construct the exact MAS or its polyhedral approximations are known (see e.g., Gutman and Cwikel (1986); Gilbert and Tan (1991); Kerrigan (2000); Blanchini and Miani (2015)), these algorithms have no guarantee of finite-time convergence except if certain stability criteria are satisfied. A sufficient condition for finite time termination requires the sets $\mathcal{X}_i$, $\mathcal{U}_i$, and $\mathcal{W}_i$ to be bounded and the closed loop system to be asymptotically stable. Computing separable controlled invariant sets in low-dimensional spaces with additive bounded disturbances overcomes the computational burden of large-scale systems (Scialanga and Ampountolas, 2017).

3. DECENTRALISED INTERPOLATING CONTROL

In the proposed decentralised approach, the inner control for each subsystem is defined in the robust maximal admissible set Ω_i for a given feedback control high-gain matrix K_i, $\forall\, i \in \mathcal{N}$. The outer control for each subsystem is defined in the robust controllable invariant set Ψ_i, $\forall\, i \in \mathcal{N}$. The set Ψ_i, $i \in \mathcal{N}$, can be obtained in an extended state and control space as the M-step robust controllable set if M is maximal, i.e., if $P_i^{M+1} = P_i^M$, $\forall\, i \in \mathcal{N}$, similarly to Nguyen et al. (2014) for the centralised control case. Alternatively, the maximal robust controllable invariant set can be computed with a λ-contractive algorithm, see Theorem 3.2 in Blanchini (1994). In general, the maximal P_i^M is not equal to the maximal robust controllable invariant set. The set $\Psi_i \setminus P_i^M$ contains all the initial states for which all future state trajectories remain inside Ψ_i but they cannot be driven to the robust maximal admissible set Ω_i. Alternatively, the set Ψ_i for each subsystem $i \in \mathcal{N}$ can be obtained by solving a semi-definite optimisation problem with linear matrix inequalities that maximises the trace of an invariant ellipsoid, associated with a low-gain controller $u_i^1(k) = -K_i^1 x_i(k)$, $i \in \mathcal{N}$ and local polyhedral constraints (3), (4). It should be noted that the global low-gain law with control gains K_i^1, $i \in \mathcal{N}$, is different from the local controller (2). In each subsystem, the interpolation between the two controls (inner and outer) is performed to make any initial state $x_i \in \Psi_i$ to enter the MAS Ω_i, $i \in \mathcal{N}$, rapidly without violating the system constraints.

Fact 4. To apply decentralised interpolating control to system (5), interconnections are considered while computing the invariant sets. Intensity of coupling between subsystems is uncertain and depends on e_{ij}. The proposed approach guarantees stability for any value of $e_{ij} \in [0, 1]$, $j \in \mathcal{N}_i$, $i \in \mathcal{N}$, i.e., any uncertainty in the coupled states.

Fig. 1 illustrates the interpolation concept in a two-dimensional state space $\mathcal{X}_i$, where the set Ψ_i depicted in yellow and the MAS Ω_i depicted in red. Suppose that any known state $x_i(k) \in \Psi_i$ can be decomposed as follows

$$x_i(k) = s_i(k) x_i^m(k) + (1 - s_i(k)) x_i^0(k), \quad i \in \mathcal{N} \tag{6}$$

where $x_i^0(k) \in \Omega_i$ and $x_i^m(k)$ is such that there exists a control $u_i^1(k) \in \mathcal{U}_i$ defined in the outer set such that

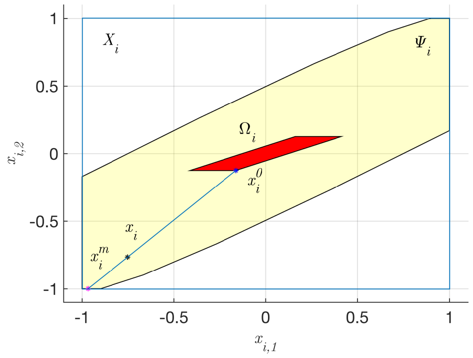

Fig. 1. In each subsystem $i \in \mathcal{N}$, any state $x_i(k)$ can be decomposed as a convex combination of $x_i^0(k) \in \Omega_i$ and $x_i^m(k) \in \Psi_i$, $i \in \mathcal{N}$.

$A_i x_i^m(k) + B_i u_i^1(k) + w_i(k) \in \Psi_i$, $\forall\, w_i \in \mathcal{W}_i$; and $s_i(k) \in [0,1]$ is an interpolating coefficient. Similarly, the control in each subsystem is decomposed as follows

$$u_i(k) = s_i(k) u_i^1(k) + (1 - s_i(k)) u_i^0(k), \quad i \in \mathcal{N} \tag{7}$$

where $u_i^0(k) = -K_i^0 x_i^0(k)$ is the inner controller (2) of each subsystem $\mathcal{S}_i$, $i \in \mathcal{N}$, and u_i^1 is the outer control. For the interpolation (6), (7), only $x_i(k) \in \Psi_i$ in each subsystem $i \in \mathcal{N}$ is known (the current state of the system). The interpolating vector consisting of coefficients s_i, state vectors $x_i^0 \in \Omega_i$ and $x_i^m \in \Psi_i$, and the outer control vector u_i^1 are all unknown and under-determination. The inner control u_i^0 is known from (2) for given $x_i^0(k)$.

The goal of control is to steer $x_i(k) \in \Psi_i$ as close as possible to the robust positively invariant set Ω_i, i.e. to minimise the local interpolating coefficients s_i, $\forall\, i \in \mathcal{N}$. Clearly, the local controller can steer the system to the origin by definition, if $s_i = 0$, $\forall\, i \in \mathcal{N}$. To solve this interpolation problem, similarly to Nguyen et al. (2014) for the case of centralised control, the following optimisation problem is formulated for each subsystem $i \in \mathcal{N}$, $\forall\, k$:

$$s_i^*(x_i) = \min_{s_i, x_i^0, x_i^m, u_i^1} s_i$$

subject to:

$$\begin{cases} F_i^0\, x_i^0 \leq g_i^0 \\ F_i^1\left(A_i\, x_i^m + B_i\, u_i^1\right) \leq g_i^1 - \max\limits_{w_i \in \mathcal{W}_i} F_i^1 w_i \\ s_i\, x_i^m + (1 - s_i)\, x_i^0 = x_i \\ 0 \leq s_i \leq 1, \quad u_i^1 \in \mathcal{U}_i. \end{cases} \tag{8}$$

This is a bilinear optimisation problem that can be transformed into a linear programming (LP) problem by setting $r_i^0 = (1 - s_i)\, x_i^0$, $r_i^m = s_i\, x_i^m$, and $v_i^1 = s_i\, u_i^1$. It follows that $r_i^0 \in (1 - s_i)\, \Omega_i$, $r_i^m \in s_i\, \Psi_i$ and $v_i^1 \in s_i\, \mathcal{U}_i$. The LP problem for each subsystem $i \in \mathcal{N}$ at each discrete time k reads:

$$s_i^*(x_i) = \min_{s_i, r_i^m, v_i^1} s_i$$

subject to: (9)

$$\begin{cases} F_i^0\, r_i^m \geq F_i^0 x_i - (1 - s_i)\, g_i^0 \\ F_i^1\left(A_i\, r_i^m + B_i\, v_i^1\right) \leq s_i \left(g_i^1 - \max\limits_{w_i \in \mathcal{W}_i} F_i^1 w_i\right) \\ 0 \leq s_i \leq 1, \quad v_i^1 \in s_i\, \mathcal{U}_i. \end{cases}$$

The second inequality in the optimisation problem guarantees that the state $x_i^m(k)$ is robust controllable by u_i^1, i.e., $A_i\, x_i^m(k) + B_i\, u_i^1 + w_i(k) \in \Psi_i$, for all $w_i \in \mathcal{W}_i$. Summarising, for each subsystem $i \in \mathcal{N}$ both Ω_i and Ψ_i are determined off-line while only the interpolation between them is performed on-line. For the interpolation the LP problem (9) is solved on-line at each time step k and its solution is denoted by $s_i^*, r_i^{m*}, v_i^{1*}$, while $r_i^{0*} = x_i - r_i^{m*}$, $i \in \mathcal{N}$. The control in each subsystem can be then recovered from (7), provided the change of variables to convert (8) to (9). Note that each LP problem (9) involves length $\left(r_i^0\right) \triangleq n_i$, $i \in \mathcal{N}$, less variables and corresponding n_i less equality constraints compared to the one proposed in Nguyen et al. (2014) for centralised interpolating control. Thus each iteration of the overall interpolating scheme is less computationally expensive and appropriate for real-time control of large-scale systems.

3.1 Decoupled subsystems

Consider the case where the dynamical system (1) is composed of (or can be decomposed into) distinct dynamical subsystems that can be independently controlled and state structural constraints are absent, i.e. $\mathcal{N}_j = \emptyset$, $\forall\, j \in \mathcal{N}$. The decoupled system reads:

$$\mathcal{S}_i : x_i(k+1) = A_i x_i(k) + B_i u_i(k), \quad i \in \mathcal{N}, \tag{10}$$

where local state and control vectors are subject to (3). The advantage here is that the inner invariant sets Ω_i and the outer invariant sets Ψ_i, $i \in \mathcal{N}$ are not "robust" invariant, so the computation of the sets is less expensive. The LP problem to solve for each subsystem $i \in \mathcal{N}$ reads:

$$s_i^*(x_i) = \min_{s_i, r_i^m, v_i^1} s_i$$

subject to:

$$\begin{cases} F_i^0\, r_i^m \geq F_i^0\, x_i - (1 - s_i)\, g_i^0 \\ F_i^1\left(A_i\, r_i^m + B_i\, v_i^1\right) \leq s_i\, g_i^1 \\ 0 \leq s_i \leq 1, \quad v_i^1 \in s_i\, \mathcal{U}_i. \end{cases} \tag{11}$$

Theorem 5. The decentralised interpolation (6), (7), (11) guarantees recursive feasibility and asymptotic stability for each subsystem $\mathcal{S}_i$ in (10), for all $x_i \in \Psi_i$, $i \in \mathcal{N}$.

Proof. Omitted. The subsystems are decoupled and thus the proof is similar to Theorem 2 in Nguyen et al. (2014).

We provide now proof of recursive feasibility and asymptotic stability of the joint system $\mathcal{S} = \bigcup_{i \in \mathcal{N}} \mathcal{S}_i$ with decentralised controls. To start with, define the vectors (for clarity the discrete time index k is omitted)

$$r^0 = \left[r_1^{0\top}\ r_2^{0\top}\ \cdots\ r_N^{0\top}\right]^\top, \quad r^m = \left[r_1^{m\top}\ r_2^{m\top}\ \cdots\ r_N^{m\top}\right]^\top,$$

$$v^0 = \left[v_1^{0\top}\ v_2^{0\top}\ \cdots\ v_N^{0\top}\right]^\top, \quad v^1 = \left[v_1^{1\top}\ v_2^{1\top}\ \cdots\ v_N^{1\top}\right]^\top.$$

Then the global state and control vectors can be decomposed as follows:

$$x(k) = r^0(k) + r^m(k), \qquad u(k) = v^0(k) + v^1(k), \tag{12}$$

where $v_i^0 = (1 - s_i)\, u_i^0$, $i \in \mathcal{N}$.

Theorem 6. The decentralised control law (12) guarantees recursive feasibility and asymptotic stability for the overall system $\mathcal{S} = \bigcup_{i \in \mathcal{N}} \mathcal{S}_i$, where $\mathcal{S}_i$ given by (10), with state constraints $\mathcal{X} = \prod_{i \in \mathcal{N}} \mathcal{X}_i$ and control constraints $\mathcal{U} = \prod_{i \in \mathcal{N}} \mathcal{U}_i$, for all $x \in \Psi$, $\Psi = \prod_{i \in \mathcal{N}} \Psi_i$.

Proof. The decomposition of state and control (12) is admissible and follows from (6), (7).

Recursive feasibility: Proof of recursive feasibility follows from the proof of the previous theorem.

Asymptotic stability: Let $s(x) = [s_1(x_1) \cdots s_N(x_N)]^\mathsf{T}$ be the vector Lyapunov function, where $s_i(x_i)$ is the Lyapunov function of the subsystem $\mathcal{S}_i$. Following the procedure in Šiljak (1991), we define the following non-negative function $V\colon \mathbb{R}^n \to \mathbb{R}^+$ for all $x \in \Psi$ as $V(x) = d^\mathsf{T} s$, where d is an all-ones vector of dimension N, so $V(x) = \sum_{i\in\mathcal{N}} s_i(x_i)$. Using the asymptotic stability of the subsystems, we know that $s_i^*(k+1) \le s_i^*(k)$ for all $i \in \mathcal{N}$. It follows that $V(x)$ is a non-increasing function and $\sum_{i\in\mathcal{N}} s_i^*(x_i)(k+1) \le \sum_{i\in\mathcal{N}} s_i^*(x_i)(k)$. The state vector x reaches (element-wise) any positively invariant sets Ω_i in finite time. Note that $s^*(x)$ is null inside Ω and the inner feedback control law $v^0(k) \triangleq u(k) = \operatorname{diag}(-K_1, \ldots, -K_N)\, x(k)$ is feasible and contractive.

3.2 Subsystems with state structural constraints

Consider now the case where the dynamic system is input-decentralised but coupled state constraints are present. Suppose that each subsystem $i \in \mathcal{N}$ can exchange some information with only a subset of neighbour subsystems, i.e., $j \in \mathcal{N}_i$, as in (1). State structural constraints can be bounded, as shown in Section 2 and the overall system can be given by (5) with additive disturbances. The constrained control problem now reads: regulate to the origin of the input-decentralised system (5) subject to constraints (3), (4). The decentralised approach aims to stabilise the overall system $\mathcal{S} = \bigcup_{i\in\mathcal{N}} \mathcal{S}_i$ by controlling each subsystem $\mathcal{S}_i$ with a control that depends only on local state vector x_i, $i \in \mathcal{N}$. It produces a closed-loop system that robustly stabilises the overall system despite failure in couplings between subsystems.

Let Ω_i and Ψ_i be the robust positively invariant set for the inner control (2) and the robust controllable invariant set (e.g., the set P^M) for the outer control, respectively. Let (12) be the decomposition of the overall state and control. We prove recursive feasibility and robust stability for the overall system $\mathcal{S} = \bigcup_{i\in\mathcal{N}} \mathcal{S}_i$ despite the influence of additive disturbances.

Theorem 7. (Recursive feasibility). The decentralised interpolation problem (6), (7), (9) guarantees recursive feasibility for the overall system (5) with state constraints $\mathcal{X} = \prod_{i\in\mathcal{N}} \mathcal{X}_i$, control constraints $\mathcal{U} = \prod_{i\in\mathcal{N}} \mathcal{U}_i$, and disturbance constraints $\mathcal{W} = \prod_{i\in\mathcal{N}} \mathcal{W}_i$, for all $x \in \Psi = \prod_{i\in\mathcal{N}} \Psi_i \subseteq \mathbb{R}^n$.

Proof. For recursive feasibility, we have to prove that $u(k)$ is in $\mathcal{U}$ and $x(k+1) \in \Psi$, for all $k \ge 0$. Since controls are independent, it is sufficient to prove that $u_i(k) \in \mathcal{U}_i$.

$$\begin{aligned} F_{u_i} u_i(k) &= F_{u_i}\{s_i(k)\, u_i^1(k) + (1 - s_i(k))\, u_i^0(k)\} \\ &= s_i(k)\, F_{u_i}\, u_i^1(k) + (1 - s_i(k))\, F_{u_i}\, u_i^0(k) \\ &\le s_i(k)\, g_{u_i} + (1 - s_i(k))\, g_{u_i} = g_{u_i}. \end{aligned}$$

Since we consider local states and controls, it is sufficient to prove that $x_i(k+1) \in \Psi_i$, for all $i \in \mathcal{N}$

$$\begin{aligned} x_i(k+1) &= A_i\, x_i(k) + B_i\, u_i(k) + w_i \\ &= A_i\, \big(s_i(k)\, x_i^m(k) + (1 - s_i(k))\; x_i^0(k)\big) \\ &\quad + B_i\, \big(s_i(k)\, u_i^1(k) + (1 - s_i(k))\; u_i^0(k)\big) + w_i(k) \\ &= s_i(k)\, \big(A_i\, x_i^m(k) + B_i\, u_i^1(k) + w_i(k)\big) \\ &\quad + (1 - s_i(k))\, \big(A_i\, x_i^0(k) + B_i\, u_i^0(k) + w_i(k)\big)\,. \end{aligned}$$

Since $A_i\, x_i^m(k) + B_i\, u_i^1(k) + w_i(k) \in \Psi_i$ and $A_i\, x_i^0(k) + B_i\, u_i^0(k) + w_i(k) \in \Omega_i \subseteq \Psi_i$, it follows that $x_i(k+1) \in \Psi_i$, for all $i \in \mathcal{N}$.

Theorem 8. (Robust stability with additive disturbances). The decentralised interpolation problem (6), (7), (9) guarantees robust asymptotic stability for the overall system (5) with state constraints $\mathcal{X} = \prod_{i\in\mathcal{N}} \mathcal{X}_i$, control constraints $\mathcal{U} = \prod_{i\in\mathcal{N}} \mathcal{U}_i$, and disturbance constraints $\mathcal{W} = \prod_{i\in\mathcal{N}} \mathcal{W}_i$, for all $x \in \Psi = \prod_{i\in\mathcal{N}} \Psi_i \subseteq \mathbb{R}^n$.

Proof. With a similar argument as in Theorem 6, define the vector Lyapunov function $V(x) = \sum_{i\in\mathcal{N}} s_i(x_i)$. Since $s_i^*(k+1) \le s_i^*(k)$ for all $i \in \mathcal{N}$, it follows that $V(x)$ is a non-increasing function and $\sum_{i\in\mathcal{N}} s_i^*(x_i)(k+1) \le \sum_{i\in\mathcal{N}} s_i^*(x_i)(k)$. The state x reaches (element-wise) any robust positively invariant sets Ω_i, i.e., $s_i = 0$, in finite time; the inner control u^0 robustly stabilises the system.

4. NUMERICAL EXAMPLES

This section demonstrates the effectiveness of the proposed decentralised interpolating control scheme for constrained systems to a decoupled system composed by two (2) subsystems and an interconnected system composed by six (6) subsystems. Three different control methods were considered and compared, namely decentralised IC (dIC), centralised IC (cIC) as in Nguyen et al. (2014), and centralised MPC (cMPC). cIC and dIC were computed by the Interpolating Control Toolbox (ICT), a Matlab toolbox recently developed by Scialanga and Ampountolas (2017), which relies on the Invariant Set toolbox developed by Kerrigan (2000). MPC was computed by the Multi-Parametric Toolbox (MPT) (Herceg et al., 2013).

4.1 Example 1: Decoupled systems

As a first example, consider the fourth-order continuous-time system with two input variables in Veillette et al. (1992). The discrete-time system matrices have been obtained assuming zero-order hold sampling at the inputs with a sampling period equal to 0.1. To obtain an input-decentralised decoupled system with distinct inputs, the controllable canonical form has been calculated. The final decoupled system consists of two subsystems $\mathcal{S} = \{\mathcal{S}_1, \mathcal{S}_2\}$ with state vector $x^\mathsf{T} = [x_1^\mathsf{T}\ x_2^\mathsf{T}]$, where $x_1, x_2 \in \mathbb{R}^2$, and input vector $u^\mathsf{T} = [u_1\ u_2]$ with u_1, u_2 being scalars. The state and control matrices are given by:

$$A_1 = \begin{bmatrix} 0 & 1 \\ -0.7960 & 1.8149 \end{bmatrix}, \quad B_1 = \begin{bmatrix} 0 \\ 1 \end{bmatrix},$$
$$A_2 = \begin{bmatrix} 0 & 1 \\ -0.7688 & 1.6578 \end{bmatrix}, \quad B_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}.$$

Local state and control variables are subject to constraints:

$$|x_{i,j}| \le 1, \quad |u_i| \le 0.1, \quad i = 1, 2, \tag{13}$$

where $x_{i,j}$ are elements of x_i, i.e., $x_i = [x_{i,1}\ x_{i,2}]^\mathsf{T}$.

In dIC, the set Ω_i for each subsystem $i = 1, 2$ is associated with a local high-gain controller (2) and the local polyhedral constraints (13). The corresponding feedback control laws are computed with weighting matrices $Q_d = I_2$ and $R_d = 10^{-5}$. The sets Ψ_1 and Ψ_2 are computed as the M-steps invariant sets P_1^{10} and P_2^{8}, respectively. Figs 2(a)

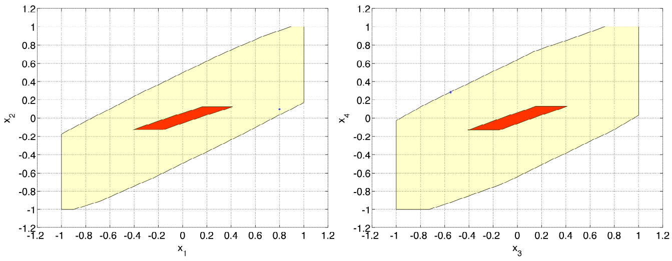

Fig. 2. Example 1: (a) Invariant sets of $\mathcal{S}_1$; yellow set is the M-step invariant set P_1^{10}. The red region is the MAS for $u_1^0 = -K_1^0 x_1^0$; (b) Invariant sets of $\mathcal{S}_2$; yellow set is the M-step invariant set P_2^8. The red region is the MAS for $u_2^0 = -K_2^0 x_2^0$.

and 2(b) show the computed invariant sets for $\mathcal{S}_1$ and $\mathcal{S}_2$, respectively. To ensure fair and comparable results, cIC and cMPC (with optimisation horizon equal to 5) are designed with the same weighting matrices ($Q_c = I_4$ and $R_c = 10^{-5} \times I_2$) and constraints (13). For cIC, the MAS Ω is computed with respect to (13) and gain matrix

$$K_{\text{cIC}} = \begin{bmatrix} -0.7960 & 1.8149 & 0 & 0 \\ 0 & 0 & -0.7688 & 1.6578 \end{bmatrix},$$

while Ψ is computed as the M-step invariant set P^{10}. Fig. 3 depicts the obtained state and control trajectories for the initial state $x_0^\mathsf{T} = [0.7986\ 0.099\ -0.552\ 0.28]$. As can be seen, the state and control trajectories with dIC are closer to cMPC than those of cIC. dIC indicates smooth and fast convergence to the equilibrium compared to cIC. Fig. 3 (right subfigure) depicts the interpolating coefficients for cIC and dIC. As expected, $s(k)$ for cIC and $s_i(k)$ for dIC are Lyapunov functions that certify the system's stability.

4.2 Example 2: System with state structural constraints

The second example concerns a high-order interconnected system with coupled state constraints (Narendra et al., 2006). The system is composed of six (6) interconnected subsystems with twelve (12) states and six (6) inputs. Each subsystem involves two states and a single input as follows:

$$A_1 = A_3 = A_5 = \begin{bmatrix} 0.2 & 1.0 \\ 0.2 & 0.5 \end{bmatrix}, A_2 = A_4 = A_6 = \begin{bmatrix} -0.5 & -0.5 \\ 0.5 & -0.25 \end{bmatrix}$$

$$B_1 = B_3 = B_5 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \quad B_2 = B_4 = B_6 = \begin{bmatrix} 2 \\ 0 \end{bmatrix}.$$

Let $E_a = \left(\begin{smallmatrix} a & 0 \\ 0 & 0 \end{smallmatrix}\right)$ and $E_b = \left(\begin{smallmatrix} b & 0 \\ 0 & 0 \end{smallmatrix}\right)$ be state structural matrices, where $a = 0.005$ and $b = 0.004$ are the intensity of interconnections. The interconnection matrices $\bar{A}_{ij}$ are:

$$\begin{aligned} \bar{A}_{1,j} &= E_a, \quad j \in \mathcal{N}_1, & \bar{A}_{2,j} &= E_b, \quad j \in \mathcal{N}_2, \\ \bar{A}_{3,j} &= E_a, \quad j \in \mathcal{N}_3, & \bar{A}_{4,j} &= E_b, \quad j \in \mathcal{N}_4, \\ \bar{A}_{5,j} &= E_a, \quad j \in \mathcal{N}_5, & \bar{A}_{6,j} &= E_b, \quad j \in \mathcal{N}_6, \end{aligned}$$

where $\mathcal{N}_i = \bigcup_{j=1 (j \neq i)}^{6} \{j\}$, $\forall\, i \in \mathcal{N}$; $e_{ij} = 1$, $\forall\, i, j$. State and control variables are subject to:

$$|x_{i,j}| \leq 6, \quad |u_i| \leq 0.4, \quad i = 1, \ldots, 6, \ j = 1, 2, \tag{14}$$

where $x_{i,j}$ are elements of $x_i = [x_{i,1}\ x_{i,2}]^\mathsf{T}$.

For the proposed dIC, the local high-gain feedback control laws are computed with weighting matrices $Q_d = I_2$ and $R_d = 10^{-5}$. The MAS Ω_i is then computed with respect to (14). The outer invariant sets Ψ_i, $i = 1, 2, \ldots, 6$, are computed as the maximal robust control invariant sets. cIC and cMPC are designed with weighting matrices ($Q_c = I_{12}$ and $R_c = 10^{-5} I_6$) and constraints (14). For cIC, the MAS Ω is computed with respect to (14) and a feedback gain matrix with Q_c and R_c above. The Ψ is computed as the maximal robust control invariant set for the overall system. The optimisation horizon of MPC is set to 10.

Table 1. Computation time with Matlab tic/toc. Processor: 2.3GHz Dual-core Intel i5.

Control method	cMPC	cIC	dIC
CPU-seconds	3.6495	1.7569	1.6412

Figs 4 and 5 depict the evolution of states and controls in the six subsystems for the initial state $x_0 = [z_1\ z_2\ z_3]^\mathsf{T}$ that belongs to the outer invariant set, where $z_1 = [-2.141\ -5.594\ -2.5\ 5.9]$, $z_2 = [3.216\ 5.38\ 5.32\ -3.493]$, and $z_3 = [-6\ 6\ -6\ -6]$. As can be seen, all three control methods have stabilised the system around the origin, albeit with slightly different control actions. Fig. 5 indicates the improvement of the proposed dIC scheme over the cIC approach and the value of decentralised interpolation in local topologies and invariant sets. Precisely, dIC indicates similar control behaviour and effort to cMPC. Thus dIC is seen to achieve the same performance as cMPC without considering an explicit quadratic cost criterion.

Fig. 5 (right subfigure) shows the interpolating coefficients for cIC and the 6 subsystems of dIC. Clearly, all coefficients are positive and non-increasing Lyapunov functions, and thus guarantee overall system's stability. Note that $\sum_{i \in \mathcal{N}} s_i(k)$ for dIC not necessarily equals to $s(k)$ for cIC. Remarkably, dIC allows for the better exploitation of the signal space with flexible interpolating coefficients and offers fast convergence. Compared to cIC, dIC allows certain subsystems i to enter their Ω_i much earlier and steer them to equilibrium; and, thus to achieve quicker their user-chosen local performance. Table 1 benchmarks the performance of the proposed dIC approach over cIC and cMPC. As can be seen, both dIC and cIC are around 50% faster than cMPC given that, in each sampling instant, IC involves the solution of an LP while MPC calls for the solution of a QP. The computational improvements of dIC will be likely to be much higher for large-scale systems that can be decomposed into distinct controlled subsystems.

5. CONCLUSIONS

This work presented a new framework for the decentralised control problem of large-scale interconnected linear systems with local state and control constraints. A distributed interpolation scheme is developed for the interpolation between a local high-gain controller with a global low-gain controller. A low-dimensional LP problem is solved on-line for each subsystem at each time step. This LP formulation involves for each subsystem $i \in \mathcal{N}$, n_i less variables and corresponding n_i less equality constraints compared to previous works. Thus, the proposed interpolation is computationally inexpensive and appropriate for real-time control of large-scale systems. Proofs of recursive feasibility and asymptotic stability of the decentralised IC scheme are given for decoupled as well as interconnected subsystems with couplings. The numerical examples demonstrated that dIC outperforms the improved centralised IC scheme (Nguyen et al., 2014) in terms of control behaviour and convergence. Note that IC is not optimal control in the sense that no cost criterion is literally assumed, which

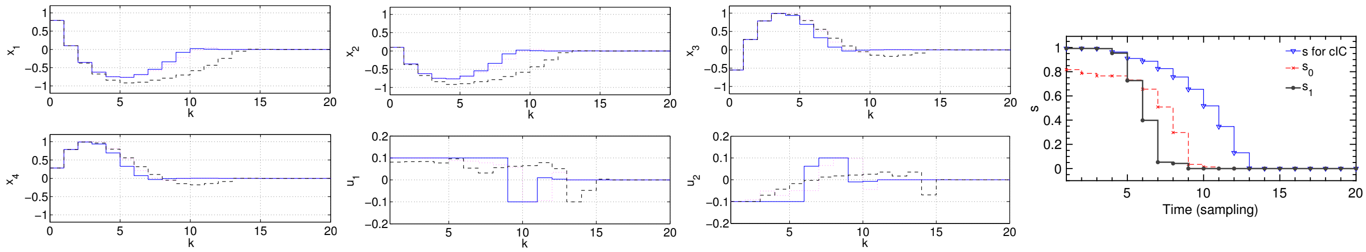

Fig. 3. Example 1. State and control trajectories for dIC (dotted magenta), cIC (dashed black) and MPC (solid blue). Interpolating coefficients.

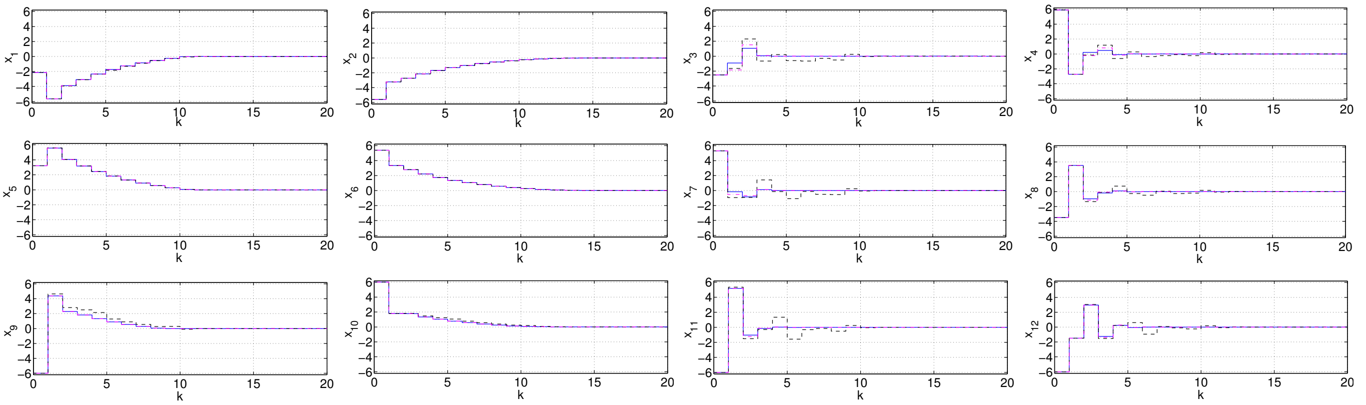
Fig. 4. Example 2. State trajectories for dIC (dot-dashed magenta), cIC (dashed black), and cMPC (solid blue).

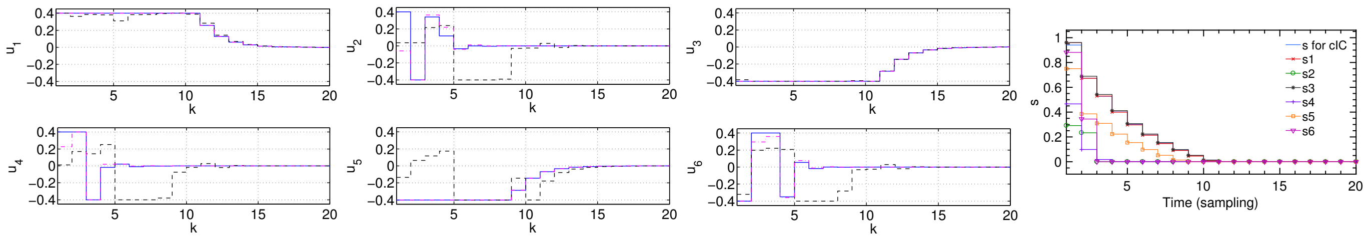

Fig. 5. Example 2. Control trajectories for dIC (dot-dashed magenta), cIC (dashed black), and cMPC (solid blue). Interpolating coefficients.

explains this counterintuitive result (see also Section 4.2). Finally, dIC is seen to be faster than MPC, while it provided almost identical control behaviour and performance.

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